SPECIAL UNIPOTENT REPRESENTATIONS WITH HALF INTEGRAL INFINITESIMAL CHARACTERS

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ABSTRACT. For any special nilpotent orbit, let $\frac{1}{2}h^\vee$ be one half of the semisimple element of a Jacobson-Morozov triple associated to the orbit. In [BV], Barbasch and Vogan defined the notion of special unipotent representations with infinitesimal character $(\frac{1}{2}h^\vee, \frac{1}{2}h^\vee)$. Some properties of such representations were discovered when $\frac{1}{2}h^\vee$ is integral, and it was stated that the same properties hold without the integrality condition. In this short note, we give a complete proof of these properties when $\frac{1}{2}h^\vee$ is not integral.

1. INTRODUCTION

Let G complex simple Lie group. For each $\lambda \in \mathfrak{h}^*$ in the dual of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is a unique maximal primitive ideal in $I_\lambda \subset U(\mathfrak{g})$ such that $I_\lambda \cap Z(U(\mathfrak{g})) = \ker(\chi_\lambda : Z(U(\mathfrak{g})) \to \mathbb{C})$, where χ_λ is the central character of the universal enveloping algebra $U(\mathfrak{g})$ corresponding to λ.

Treating G as a real Lie group with maximal compact subgroup K, we identify $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \times \mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}_\mathbb{C} = \mathfrak{h} \times \mathfrak{h}$, compact torus $\mathfrak{t} = \{(x, -x) \in \mathfrak{h}_\mathbb{C} | x \in \mathfrak{h}\}$ and split torus $\mathfrak{a} = \{(x, x) \in \mathfrak{h}_\mathbb{C} | x \in \mathfrak{h}\}$ with $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$. Given $(\lambda, \mu) \in \mathfrak{h}_\mathbb{C}^*$ so that $(\lambda - \mu)$ is a weight of a finite dimensional holomorphic representation of G, then the *principal series representation* with character (λ, μ) is the $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-module

$$X(\lambda, \mu) = K - \text{finite part of } Ind_B^G(e^{(\lambda,\mu)} \boxtimes \text{triv}).$$

For each $\lambda \in \mathfrak{h}^*$ and $w \in W$, one would like to study which irreducible quotient $\overline{X}(\lambda, w\lambda)$ of $X(\lambda, w\lambda)$ satisfies $LAnn_{U(\mathfrak{g})}\overline{X}(\lambda, w\lambda) = RAnn_{U(\mathfrak{g})}\overline{X}(\lambda, w\lambda) = I_\lambda$. These are the irreducible modules with the smallest possible associated variety. In particular, the spherical module $\overline{X}(\lambda, \lambda) = U(\mathfrak{g})/I_\lambda$ satisfies the above property.

As hinted in [BV], there are certain values of λ that are of special interest in representation theory – Let $\mathcal{O}^\vee$ be a special nilpotent in the Langlands dual ${}^L\mathfrak{g}$, and $\frac{1}{2}h^\vee \in {}^L\mathfrak{h} \cong \mathfrak{h}^*$ be one half of the semisimple element of a Jacobson-Morozov triple associated to $\mathcal{O}^\vee$. We would like to study irreducible $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-modules with $\lambda = \lambda_{\mathcal{O}^\vee} := \frac{1}{2}h^\vee$.

There are several reasons why irreducible modules with infinitesimal character $(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ are of particular interest, and we state a couple of them here. Firstly, it is related to the classification of spherical unitary spectrum for reductive Lie groups over $\mathbb{R}$, $\mathbb{C}$ or p-adic fields. When the base field is $\mathbb{C}$, Barbasch in [B1] showed that $\overline{X}(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ is unitary for all classical $\mathcal{O}^\vee$. More generally, [B3] showed that a similar theorem holds for spherical modules in split real and p-adic reductive Lie groups. In both cases, spherical modules with infinitesimal character $\lambda_{\mathcal{O}^\vee}$ are the 'building blocks' of the spherical unitary spectrum.

It is worth noting that for the p-adic case, the unitarity of spherical modules with infinitesimal character $\lambda_{\mathcal{O}^\vee}$ comes naturally from the Kazhdan-Lusztig classification of graded Hecke algebra modules. Consequently, its unitarity holds for exceptional Lie groups as well. Whereas in the real and complex classical Lie groups, a lot of arguments involving petite K-types, intertwining operators and associated varieties are required to prove the unitarity of the spherical modules. This naturally brings us to the question on the unitarity of $\overline{X}(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ for exceptional $\mathcal{O}^\vee$.

The second reason is related to the idea of quantization. Roughly speaking, quantization attaches a (preferably unitarizable) $(\mathfrak{g}, K)$-module π to a local system of a nilpotent orbit $\mathcal{O}$. More details can be found in [W1] and [W2]. In particular, the associated variety of π must be equal to $AV(\pi) = \overline{\mathcal{O}}$.

We now recall some known results about the above quantization scheme. In [BV], the *special unipotent representations* are defined as follows:

Definition 1.1 ([BV], Definition 1.17)**.** *Let $\mathcal{O}^\vee$ be a special nilpotent orbit and $2\lambda_{\mathcal{O}^\vee} := h^\vee$ is the semisimple element in a Jacobson-Morozov triple attached to $\mathcal{O}^\vee$. Writing $\mathcal{O}$ as the Lusztig-Spaltenstein dual of $\mathcal{O}^\vee$, then the* **special unipotent representations** *attached to $\mathcal{O}^\vee$ are given by the set*

$$\Pi(\mathcal{O}^\vee) := \{\overline{X}(\lambda_{\mathcal{O}^\vee}, w\lambda_{\mathcal{O}^\vee}) \mid AV(\overline{X}(\lambda_{\mathcal{O}^\vee}, w\lambda_{\mathcal{O}^\vee})) \subseteq \overline{\mathcal{O}}\}.$$

Note that in [W1] *or* [W2]*, such representations are attached to $\mathcal{O}$ rather than $\mathcal{O}^\vee$.*

Assuming $\mathcal{O}^\vee$ is classical and smoothly cuspidal, then by [B4] elements in $\Pi(\mathcal{O}^\vee)$ give quantization models for *all* local systems of $\mathcal{O}$. In particular, $\overline{X}(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ is a unitarizable $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module such that $\overline{X}(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee}) \cong R(\mathcal{O})$.

The main results in [BV] were proved under the condition that $\lambda_{\mathcal{O}^\vee}$ integral, i.e. $\mathcal{O}^\vee$ is even, where the representations in $\Pi(\mathcal{O}^\vee)$ are called *integral* special unipotent representations. In this case, they proved that elements in $\Pi(\mathcal{O}^\vee)$ must have associated varieties equal to $\overline{\mathcal{O}}$. Also, the number of irreducible modules $\overline{X}(\lambda_{\mathcal{O}^\vee}, w\lambda_{\mathcal{O}^\vee}) \in \Pi(\mathcal{O}^\vee)$ is equal to

the number of irreducible representations of the Lusztig quotient $\overline{A}(\mathcal{O}^\vee)$. Their character formulas are also written down explicitly.

The work of [B4] and [W2] rely on the fact that all properties of integral special unipotent representations proved in [BV] hold for special unipotent representations as well. However, this was only stated (but not proved) in the last paragraph of Section 1 in [BV]. There are some partial proofs along this direction, though. For example, in Chapter 5 of [McG], McGovern proved that all elements in $\Pi(\mathcal{O}^\vee)$ have associated variety equal to $\overline{\mathcal{O}}$ for classical $\mathcal{O}^\vee$. The idea in McGovern's proof also applies for exceptional groups. The aim of this short note is to give a complete proof of this fact, that is:

Theorem 1.2. *All elements in $\Pi(\mathcal{O}^\vee)$ have associated variety equal to $\overline{\mathcal{O}}$. Moreover, the cardinality of $\Pi(\mathcal{O}^\vee)$ is equal to the number of irreducible representations of $\overline{A}(\mathcal{O}^\vee)$, and the character formulas for all $\overline{X}(\lambda_{\mathcal{O}^\vee}, w\lambda_{\mathcal{O}^\vee}) \in \Pi(\mathcal{O}^\vee)$ are of the form given in Theorem III of* [BV].

Given that Theorem 1.2 holds, we would like to ask whether all special unipotent representations give quantization models for local systems $\mathcal{O}$ for all special exceptional $\mathcal{O}^\vee$, regardless of the integrality of $\lambda_{\mathcal{O}^\vee}$, and whether $\overline{X}(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ are unitarizable. We will deal with part of these problems in a subsequent work.

The manuscript is organized as follows. Section 2 deals with Theorem 1.2 for classical Lie groups. Section 3 deal with Theorem 1.2 for exceptional Lie groups of Type F_4, E_6, E_7 and E_8 respectively. As mentioned above, Theorem 1.2 is known to be true for all even $\mathcal{O}^\vee$, so we only study the special, non-even orbits. Since there are no such orbits in G_2, we simply skip this case.

The main tool for the proof of Theorem 1.2 is the Kazhdan-Lusztig conjecture, which states that the character theory of $(\mathfrak{g}_\mathbb{C}, K_\mathbb{C})$-modules with infinitesimal character $\lambda_{\mathcal{O}^\vee}$ can be reduced to that of the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ with roots α in $\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda_{\mathcal{O}^\vee}\rangle \in \mathbb{Z}$. Upon restricting to $\mathfrak{g}'$, one can reduce to the setting where [BV] applies.

2. Classical Lie algebras

Using Proposition 2.2 of [W2], we describe all special nilpotent orbits in ${}^L\mathfrak{g}$ of type B, C and D as follows:

- **Type** B**:** Let $\mathcal{O}^\vee = [r_{2k} \geq r_{2k-1} \geq \cdots \geq r_0]$ be a special orbit. Separate all **even** rows (which must be of the form $r_{2l-1} = r_{2l-2} = \alpha$), along with odd row pairs of

the form $r_{2l} = r_{2l-1} = \beta$ and get
$$\begin{aligned}\mathcal{O}^\vee =&[r''_{2q} > r''_{2q-1} \geq r''_{2q-2} > \cdots \geq r''_2 > r''_1 \geq r''_0] \\ &\cup [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x] \cup [\beta_1, \beta_1, \ldots, \beta_y, \beta_y],\end{aligned}$$

- **Type** C**:** Let $\mathcal{O}^\vee = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_1]$ be a special orbit. Separate all **odd** rows (which must be of the form $r_{2l-1} = r_{2l-2} = \alpha$), and even row pairs of the form $r_{2l} = r_{2l-1} = \beta$ and get
$$\begin{aligned}\mathcal{O}^\vee =&[r''_{2q+1} \geq r''_{2q} > r''_{2q-1} \geq \cdots > r''_3 \geq r''_2 > r''_1] \\ &\cup [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x] \cup [\beta_1, \beta_1, \ldots, \beta_y, \beta_y],\end{aligned}$$
- **Type** D**:** Let $\mathcal{O}^\vee = [r_{2k+1} \geq r_{2k} \geq \cdots \geq r_0]$ be a special, non-very even orbit. Separate all **even** rows (which must be of the form $r_{2l-1} = r_{2l-2} = \alpha$), and all odd row pairs $r_{2l} = r_{2l-1} = \beta$ and get
$$\begin{aligned}\mathcal{O}^\vee =&[r''_{2q+1} \geq r''_{2q} > r''_{2q-1} \geq \cdots \geq r''_2 > r''_1 \geq r''_0] \\ &\cup [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x] \cup [\beta_1, \beta_1, \ldots, \beta_y, \beta_y],\end{aligned}$$

By Proposition 2.2 of [W2], $\overline{A}(\mathcal{O}^\vee) = (\mathbb{Z}/2\mathbb{Z})^q$, regardless of the number of α_i's and β_j's. Also, a special orbit $(\mathcal{O}')^\vee$ is even iff there are no α_i's in the above expression. For such orbits, Theorem 1.2 and the results in [BV] hold.

For any special orbit $\mathcal{O}^\vee$, consider $\mathcal{O}^\vee = (\mathcal{O}')^\vee \cup [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$, where $(\mathcal{O}')^\vee$ is the orbit of the same type as $\mathcal{O}^\vee$ with no α_i's. As mentioned in the introduction, we use the Kazhdan-Lusztig conjecture to reduce our study to $\mathfrak{g}'$. More precisely, let ${}^L\mathfrak{g}'$ be the Lie subalgebra of ${}^L\mathfrak{g}$ whose roots are given by the roots $\alpha^\vee$ in ${}^L\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda_{\mathcal{O}^\vee} \rangle \in \mathbb{Z}$.

For $\mathcal{O}^\vee = (\mathcal{O}')^\vee \cup [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$, the coordinates of $\lambda_{\mathcal{O}^\vee}$ consists of:
- Type B, D: integers coming from $(\mathcal{O}')^\vee$, half-integers coming from $[\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$.
- Type C: half-integers coming from $(\mathcal{O}')^\vee$, integers coming from $[\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$.

Therefore, ${}^L\mathfrak{g}' = {}^L\mathfrak{g}'_1 + {}^L\mathfrak{g}'_2$, where
- Type B, D: ${}^L\mathfrak{g}'_1$ is of Type B, D; ${}^L\mathfrak{g}'_2$ is of Type D.
- Type C: ${}^L\mathfrak{g}'_1$ is of Type C; ${}^L\mathfrak{g}'_2$ is of Type C.

So we have reduced our study of special unipotent representations attached to $\mathcal{O}^\vee$ in ${}^L\mathfrak{g}$ to that of
$$(\mathcal{O}')^\vee \text{ in } {}^L\mathfrak{g}'_1\ ,\ [\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x] \text{ in } {}^L\mathfrak{g}'_2.$$
Both orbits are even in the respective Lie algebras, so [BV] applies. We now focus on the second orbit – in fact, $[\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$ has trivial Lusztig quotient, and the smallest associated variety attached to this orbit is equal to its Lusztig-Spaltenstein dual given by:

• Type B, D: $(\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x) = Ind_{\mathfrak{gl}(\alpha_1)+\cdots+\mathfrak{gl}(\alpha_x)}^{\mathfrak{g}_2'}(0)$ of Type D;
• Type C: $(\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x, 1) = Ind_{\mathfrak{gl}(\alpha_1)+\cdots+\mathfrak{gl}(\alpha_x)}^{\mathfrak{g}_2'}(0)$ of Type B.

Note that the above orbits, written in round brackets, are described in terms of the columns of its Young diagram. Therefore, *the* special unipotent representation attached to $[\alpha_1, \alpha_1, \ldots, \alpha_x, \alpha_x]$ must be equal to $Ind_{GL(\alpha_1)\times\cdots\times GL(\alpha_x)}^{G_2'}(\mathrm{triv})$.

Taking into account the $(\mathbb{Z}/2\mathbb{Z})^q$ special unipotent representations$X_{(\mathcal{O}')^\vee,\pi} \in \Pi((\mathcal{O}')^\vee)$, we conclude that

$$\Pi(\mathcal{O}^\vee) = \{Ind_{G_1'\times GL(\alpha_1)\times\cdots\times GL(\alpha_x)}^{G}(X_{(\mathcal{O}')^\vee,\pi} \boxtimes \mathrm{triv} \boxtimes \cdots \boxtimes \mathrm{triv}) \mid X_{(\mathcal{O}')^\vee,\pi} \in \Pi((\mathcal{O}')^\vee)\}$$

(Compare this with Equation (3) in Section 3.2 of [W2]). In particular, $|\Pi(\mathcal{O}^\vee)| = |\Pi((\mathcal{O}')^\vee)|$ $= 2^q$. But we have seen from the beginning of this section that $\overline{A}(\mathcal{O}^\vee) = (\mathbb{Z}/2\mathbb{Z})^q$, hence we have proved the second statement of Theorem 1.2.

Finally, we need to see the associated variety for the elements in $\Pi(\mathcal{O}^\vee)$ is equal to $\mathcal{O}$, the Lusztig-Spaltenstein dual of $\mathcal{O}^\vee$. Recall from Section 4.2 of [W2] that $\mathcal{O}$ is equal to

• **Type** B**:**

$$(\rho_{2q+1}'' + 1 > \rho_{2q}'' - 1 \geq \rho_{2q-1}'' + 1 > \cdots \geq \rho_3'' + 1 > \rho_2'' - 1 \geq \rho_1'' + 1) \cup \bigcup (\alpha_i, \alpha_i) \cup \bigcup (\beta_j, \beta_j).$$

• **Type** C**:**

$$(\rho_{2q}'' - 1 \geq \rho_{2q-1}'' + 1 > \rho_{2q-2}'' - 1 \geq \cdots > \rho_2'' - 1 \geq \rho_1'' + 1 > \rho_0'' - 1) \cup \bigcup (\alpha_i, \alpha_i) \cup \bigcup (\beta_j, \beta_j).$$

• **Type** D**:**

$$(\rho_{2q+1}'' + 1 > \rho_{2q}'' - 1 \geq \rho_{2q-1}'' + 1 > \cdots \geq \rho_2'' - 1 > \rho_1'' + 1 \geq \rho_0'' - 1) \cup \bigcup (\alpha_i, \alpha_i) \cup \bigcup (\beta_j, \beta_j).$$

In particular, the Lusztig-Spaltenstein dual $\mathcal{O}'$ of $(\mathcal{O}')^\vee$ is equal to the above expression without all the α_i's. Since the associated variety for each of $X_{(\mathcal{O}')^\vee,\pi}$ is equal to $\overline{\mathcal{O}'}$, the associated variety for each of the element in $\Pi(\mathcal{O}^\vee)$ is equal to the closure of

$$Ind_{\mathfrak{g}_1'+\mathfrak{gl}(\alpha_1)+\cdots+\mathfrak{gl}(\alpha_x)}^{\mathfrak{g}}(\mathcal{O}' + 0 + \cdots + 0) = \mathcal{O}' \cup \bigcup (\alpha_i, \alpha_i),$$

which is precisely equal to $\mathcal{O}$ as required.

3. Exceptional Lie algebras

Let $\mathcal{O}^\vee \subset {}^L\mathfrak{g}$ be an exceptional, special nilpotent orbit with $\lambda_{\mathcal{O}^\vee}$ not integral. As in the previous section, let ${}^L\mathfrak{g}'$ be the Lie subalgebra of ${}^L\mathfrak{g}$ whose roots are given by the roots $\alpha^\vee$ in ${}^L\mathfrak{g}$ satisfying $\langle \alpha^\vee, \lambda_{\mathcal{O}^\vee} \rangle \in \mathbb{Z}$. Then we will take a slightly different approach from the previous section to determine the special unipotent representations (c.f. Section 5 of [BV]). This method applies for all simple Lie algebras, and works especially well when

the rank is small.

Consider the subcollection of roots $\beta^\vee$ in ${}^L\mathfrak{g}'$ such that $\langle \beta^\vee, \lambda_{\mathcal{O}^\vee}\rangle = 0$. The roots form a Levi subalgebra ${}^L\mathfrak{l}'$ of ${}^L\mathfrak{g}'$. We can reduce our study of unipotent representations attached to $\mathcal{O}^\vee \subset {}^L\mathfrak{g}$ to that of $(\mathcal{O}')^\vee := Ind_{{}^L\mathfrak{l}'}^{{}^L\mathfrak{g}'}(0) \subset {}^L\mathfrak{g}'$. In F_4, E_6 or E_7, ${}^L\mathfrak{g}'$ consists only of classical factors, therefore Theorem 1.2 and the analogous statements in [BV] hold for $(\mathcal{O}')^\vee$. In E_8, ${}^L\mathfrak{g}'$ is either equal to $E_7 + A_1$ or D_8, so we can prove Theorem 1.2 for E_8 based on the validity of Theorem 1.2 for E_7.

We now describe the algorithm to compute the minimal associated variety attached to $(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$. Consider the representation σ of the Weyl group $W({}^LG')$ of ${}^LG'$ generated by the positive roots in ${}^L\mathfrak{l}'$. This is equal to the Springer representation attached to $(\mathcal{O}')^\vee$. Let $\sigma' := \sigma \otimes \mathrm{sgn}$ be the Springer representation attached to the Lusztig-Spaltenstein dual $\mathcal{O}'$ of $(\mathcal{O}')^\vee$. Then the smallest associated variety attached to $(\lambda_{\mathcal{O}^\vee}, \lambda_{\mathcal{O}^\vee})$ is the nilpotent orbit $\mathcal{O}$ whose Springer representation is equal to the truncated induction $j_{W(G')}^{W(G)}(\sigma')$. In particular, if σ' has b-value (the smallest value i such that σ' appears in the i^{th} symmetric power of V) equal to x, then $\mathcal{O}$ must be of codimension $2x$ inside the full nilpotent variety of $\mathfrak{g}$. We will show the first statement of Theorem 1.2 holds by showing $\mathcal{O}$ is equal to the Lusztig-Spaltenstein dual of $\mathcal{O}^\vee$.

To find out the number of irreducible representations showing up in $\Pi(\mathcal{O}^\vee)$, recall in (5.26)-(5.28) of [BV] that all representations $\rho_C \in V^L(w_0 w_{\mathcal{O}'})$ are parametrized by a conjugacy class $C \subset \overline{A}(\mathcal{O}')$, with $\rho_{\text{triv}} = \sigma'$. The ρ_C are precisely the irreducible representations appearing in the character formulas in Theorem III of [BV]. Therefore, $|\Pi(\mathcal{O}^\vee)|$ is equal to the number of conjugacy classes of $\overline{A}(\mathcal{O}')$. We will check case-by-case that $\overline{A}(\mathcal{O}') \cong \overline{A}(\mathcal{O}) \cong \overline{A}(\mathcal{O}^\vee)$, so that the last statement of Theorem 1.2 holds.

In the following subsections, we will list all $\mathcal{O}^\vee$ with non-integral $\lambda_{\mathcal{O}^\vee}$ and its Lusztig-Spaltenstein dual $\mathcal{O}$. We will give their corresponding $\mathfrak{g}'$, $\mathfrak{l}'$ (in terms of their types) and $\sigma' \in \hat{W}(G')$ described in the previous paragraphs. Moreover, we also give the orbit $\mathcal{O}'$ corresponding to σ'. One can check from [AL] and [L1] that $j_{W(G')}^{W(G)}(\sigma')$ is the Springer representation corresponding to $\mathcal{O}$ and $\overline{A}(\mathcal{O}) = \overline{A}(\mathcal{O}')$.

The computations are carried out by `LiE` [LiE] and `MATLAB`.

3.1. F_4**.** The results for F_4 are as follows:

$\mathcal{O}^\vee$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathfrak{l}'$	$\sigma' \in \hat{W}(G')$	$b_{\sigma'}$	$\mathcal{O}'$
$\widetilde{A}_1$	$F_4(a_1)$	B_4	B_3	$((1),(3))$	1	$[711]$
$A_1+\widetilde{A}_1$	$F_4(a_2)$	C_3+A_1	A_2+A_1	$((1),(2))\boxtimes(2)$	2	$[3^2]+[2]$
C_3	A_2	C_3+A_1	$0+A_1$	$(\phi,(1^3))\boxtimes(2)$	9	$[1^6]+[2]$

3.2. E_6. The results for E_6 are as follows:

$\mathcal{O}^\vee$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathfrak{l}'$	$\sigma' \in \hat{W}(G')$	$b_{\sigma'}$	$\mathcal{O}'$
A_1	$E_6(a_1)$	A_5+A_1	A_5+0	$(6)\boxtimes(1^2)$	1	$[6]+[1^2]$
$2A_1$	D_5	D_5	D_4	$((41),\phi)$	2	$[71^3]$
A_2+A_1	$D_5(a_1)$	A_5+A_1	A_3+0	$(41^2)\boxtimes(1^2)$	4	$[41^2]+[1^2]$
A_2+2A_1	A_4+A_1	D_5	D_2+A_2	$((2^2),(1))$	5	$[3^31]$
A_3	A_4	D_5	A_3	$((31^2),\phi)$	6	$[51^5]$
A_4+A_1	A_2+2A_1	A_5+A_1	A_1+0	$(21^4)\boxtimes(1^2)$	11	$[21^4]+[1^2]$
$D_5(a_1)$	A_2+A_1	D_5	A_1	$((1^4),(1))$	13	$[2^21^6]$

3.3. E_7. The results for E_7 are as follows:

$\mathcal{O}^\vee$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathfrak{l}'$	$\sigma' \in \hat{W}(G')$	$b_{\sigma'}$	$\mathcal{O}'$
A_1	$E_7(a_1)$	D_6+A_1	D_6+0	$((6),\phi)\boxtimes(1^2)$	1	$[11,1]+[1^2]$
$2A_1$	$E_7(a_2)$	D_6+A_1	D_5+A_1	$((51),\phi)\boxtimes(2)$	2	$[91^3]+[2]$
A_2+A_1	$E_6(a_1)$	D_6+A_1	$(D_4+A_1)+0$	$((41),(1))\boxtimes(1^2)$	4	$[731^2]+[1^2]$
A_2+2A_1	$E_7(a_4)$	D_6+A_1	$(A_3+A_2)+A_1$	$((32),(1))\boxtimes(2)$	5	$[53^31]+[2]$
A_3	$D_6(a_1)$	D_6+A_1	D_4+A_1	$((41^2),\phi)\boxtimes(2)$	6	$[71^5]+[2]$
$D_4(a_1)+A_1$	$E_6(a_3)$	D_6+A_1	$(A_3+A_1)+0$	$((31^2),(2))\boxtimes(1^2)$	8	$[531^4]+[1^2]$
A_3+A_2	$D_5(a_1)+A_1$	D_6+A_1	$(D_2+A_2)+A_1$	$((2^21),(1))\boxtimes(2)$	9	$[3^31^3]+[2]$
A_4+A_1	A_4+A_1	D_6+A_1	$4A_1+0$	$((21^2),(1^2)\boxtimes(1^2)$	11	$[3^22^21^2]+[1^2]$
$D_5(a_1)$	A_4	D_6+A_1	$3A_1+A_1$	$((21^3),(1))\boxtimes(2)$	13	$[3^21^6]+[2]$
D_5+A_1	$2A_2$	D_6+A_1	$2A_1+0$	$((21^4),\phi)\boxtimes(1^2)$	21	$[31^9]+[1^2]$
$D_6(a_1)$	A_3	D_6+A_1	A_1+A_1	$((1^5),(1))\boxtimes(2)$	21	$[2^21^8]+[2]$

3.4. E_8. The results for E_8 are as follows:

$\mathcal{O}^\vee$	$\mathcal{O}$	$\mathfrak{g}'$	$\mathfrak{l}'$	$\sigma' \in \hat{W}(G')$	$b_{\sigma'}$	$\mathcal{O}'$
A_1	$E_8(a_1)$	$E_7 + A_1$	$E_7 + 0$	$1_x \boxtimes (1^2)$	1	$E_7 + [1^2]$
$2A_1$	$E_8(a_2)$	D_8	D_7	$((71), \phi)$	2	$[13, 1^3]$
$A_2 + A_1$	$E_8(a_4)$	$E_7 + A_1$	$D_6 + 0$	$56'_a \boxtimes (1^2)$	4	$E_7(a_3) + [1^2]$
$A_2 + 2A_1$	$E_8(b_4)$	D_8	$D_5 + A_2$	$((52), (1))$	5	$[93^2 1]$
A_3	$E_7(a_1)$	D_8	D_6	$((61^2), \phi)$	6	$[11, 1^5]$
$D_4(a_1) + A_1$	$E_8(a_6)$	$E_7 + A_1$	$(A_5 + A_1) + 0$	$315'_a \boxtimes (1^2)$	8	$E_7(a_5) + [1^2]$
$A_3 + A_2$	$D_7(a_1)$	D_8	$D_4 + A_2$	$((421), (1))$	9	$[73^2 1^3]$
$A_4 + A_1$	$E_6(a_1) + A_1$	$E_7 + A_1$	$(D_4 + A_1) + 0$	$420_a \boxtimes (1^2)$	11	$D_5(a_1) + [1^2]$
$A_4 + 2A_1$	$D_7(a_2)$	D_8	$A_3 + A_2 + A_1$	$((321), (1^2))$	12	$[53^3 1^2]$
$D_5(a_1)$	$E_6(a_1)$	D_8	$D_4 + A_1$	$((41^3), (1))$	13	$[731^6]$
$D_5(a_1) + A_1$	$E_7(a_4)$	$E_7 + A_1$	$(A_3 + A_2) + A_1$	$378'_a \boxtimes (2)$	14	$(A_3 + A_2) + [2]$
$A_4 + A_2 + A_1$	$A_6 + A_1$	$E_7 + A_1$	$(A_3 + A_2 + A_1) + 0$	$210_b \boxtimes (1^2)$	14	$(A_3 + A_2 + A_1) + [1^2]$
$D_6(a_1)$	$E_6(a_3)$	D_8	$A_3 + A_1$	$((321^2), (1))$	21	$[531^8]$
$A_6 + A_1$	$A_4 + A_2 + A_1$	$E_7 + A_1$	$(A_2 + 3A_1) + 0$	$105'_b \boxtimes (1^2)$	22	$(A_2 + 3A_1) + [1^2]$
$E_7(a_4)$	$D_5(a_1) + A_1$	$E_7 + A_1$	$(A_2 + 2A_1) + A_1$	$189_b \boxtimes (2)$	22	$(A_2 + 2A_1) + [2]$
$D_7(a_2)$	$A_4 + 2A_1$	D_8	$4A_1$	$((21^4), (1^2))$	24	$[3^2 2^2 1^6]$
$E_6(a_1) + A_1$	$A_4 + A_1$	$E_7 + A_1$	$4A_1 + 0$	$120'_a \boxtimes (1^2)$	26	$(A_2 + A_1) + [1^2]$
$E_7(a_3)$	A_4	$E_7 + A_1$	$A_2 + A_1$	$56_a \boxtimes (2)$	30	$A_2 + [2]$
$E_7(a_1)$	A_3	$E_7 + A_1$	$A_1 + A_1$	$7_a \boxtimes (2)$	46	$A_1 + [2]$

References

[A] D. Alvis, *Induce/restrict matrices for exceptional Weyl groups*, `arXiv:0506377`

[AL] D. Alvis, G. Lusztig, *On Springer's correspondence for simple groups of type $E_n (n = 6, 7, 8)$*, Math. Proc. Camb. Phil. Soc. (1982) **92**, 65-72

[B1] D. Barbasch, *The unitary dual for complex classical Lie groups*, Invent. Math. **96** (1989), 103-176

[B2] D. Barbasch, *Representations with maximal primitive ideal*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), Progr. Math. **92** (1990), 317-331

[B3] D. Barbasch, *The unitary spherical spectrum for split classical groups*, Journal of the Inst. of Math. Jussieu (2010) **9**(2), 265–356

[B4] D. Barbasch, *Unipotent representations and the dual pair correspondence*, Representation Theory, Number Theory and Invariant Theory: In honor of Roger Howe on the occasion of his 70th birthday (2017)

[BV] D. Barbasch, D. Vogan, *Unipotent representations of complex semisimple Lie groups*, Ann. of Math. **121** (1985), 41-110

[Ca] R. Carter, *Finite Groups of Lie Type*, Wiley & Sons

[CM] D. Collingwood, W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Norstrand Reinhold Mathematics Series

[dGE] W. A. de Graff, A. Elashvili, *Induced Nilpotent Orbits of the Simple Lie Algebras of Exceptional Type*, Georgian Mathematical Journal **16** (2009), 257-278

[LiE] M. A. A. van Leeuwen, A. M. Cohen and B. Lisser, *LiE, A Package for Lie Group Computations*, Computer Algebra Nederland, Amsterdam (1992)

[L1] G. Lusztig, *Characters of reductive groups over a finite field*, Ann. Math. Studies **107** (1984), Princeton University Press
[L2] G. Lusztig, *Notes on unipotent classes*, Asian J. Math. **1** (1997), 194-207
[L3] G. Lusztig, *Unipotent classes and special Weyl group representations*, J. Algebra. **321** (2009), 3418-3449
[LS] G. Lusztig, N. Spaltenstein, *Induced Unipotent Classes*, J. London Math. Soc. (2) **19** (1979), no. 1, 41-52
[McG] W. M. McGovern, *Completely prime maximal ideals and quantization*, Mem. Amer. Math. Soc. **519** (1994)
[W1] K. D. Wong, *Quantization of Special Symplectic Nilpotent Orbits and Normality of their Closures*, J. Algebra **462** (2016), 37-53
[W2] K. D. Wong, *Some Explicit Calculations of the Lusztig-Vogan Bijection, and a Conjecture of Achar-Sommers*, arXiv:1606.05975